Derivation and Analysis of Piecewise Constant Conservative Approximation for Anisotropic Diffusion Problems

A. Agouzal, Naïma Debit

Derivation and Analysis of Piecewise Constant Conservative Approximation for Anisotropic Diffusion Problems

A. Agouzal [a], N. Debit [b]

[a] *Université de Lyon ; Université Lyon 1 ; CNRS, UMR5208, Institut Camille Jordan, 43 blvd du 11 novembre 1918, F-69622 Villeurbanne-Cedex, France.*

[b] *Université de Lyon ; Université Lyon 1 ; CNRS, UMR5208, Bât ISTIL, 15 Bd Latarjet, F-69622 Villeurbanne-Cedex, France.*

Abstract

A variational approach to derive a piecewise constant conservative approximation of anisotropic diffusion equations is presented. A priori error estimates are derived assuming usual mesh regularity constraints and a posteriori error indicator is proposed and analyzed for the model problem.

Key words:
1991 MSC: 65N30 Finite elements, conservative approximation, a posteriori error estimate

1 Introduction

Various phenomena in scientific fields such as geoscience, oil reservoir simulation, hydrogeology, biology ..., are generally modeled by anisotropic diffusion equations. The usual discretization schemes of this equations are finite difference, finite element or finite volume methods. The last are piecewise constant conservative approximation and are actually very popular in oil engineering, the reason probably being that complex coupled physical phenomena may be discretized on the same grids (see for instance [9] and references therein). But the well known five point on rectangles and four point schemes on triangles are not easily adapted to heterogeneous anisotropic diffusion operators, and so an enlarged stencil scheme which handles anisotropy on meshes satisfying an orthogonality property was proposed and analyzed in [2,6,7]. Let us recall that a huge literature exists in engineering study setting. However, even though these schemes perform well in the number of cases, their convergence

Email addresses: `agouzal@univ-lyon1.fr` (A. Agouzal), `Naima.Debit@univ-lyon1.fr` (N. Debit).

analysis often seems out of reach, unless some additional geometrical conditions are imposed. Moreover, actually in several applications the discretization meshes are imposed by engineering and computing considerations, therefore we have to deal with unstructured meshes.
A motivation for this work was to construct such a piecewise constant approximation for anisotropic diffusion problems which could satisfy the two assumptions : First, the resulting formulation is well-defined on general unstructured meshes, assuming usual finite element mesh regularity constraints. Secondly, the given scheme leads to standard algebraic system, for which we can use the existing efficient numerical solvers. This last point is of major importance in the coupling of physical models, from the implementation point of view and good adaptivity properties.

An outline of the paper is as follows. Among all the developments we briefly introduce the functional framework and some usual notations. In second section, we introduce the numerical scheme for the anisotropic diffusion problem and main approximation analysis results are given. A focus is made on the treatment of an additional reaction term. Finally, in section three we propose and analyze an a posteriori error indicator for the diffusion model problem.

Functional Framework and some notations

Let ω be a bounded polygonal domain of $I\!R^2$. We denote by $H^s(\omega)$ the usual Sobolev space $W^{s,2}(\omega)$ (see e.g [1]), endowed with the norm $||.||_{s,\omega}$and $H^s_0(\omega)$ is the closure of $\mathcal{D}(\omega)$ in $H^s(\omega)$. For the semi-norm, we use the notation $|.|_{s,\omega}$. We introduce the set $H(\mathrm{div},\omega)$ of vector fields $p \in (L^2(\omega))^d$ and $\mathrm{div}\, p \in L^2(\omega)$. Equipped with the norm $\|.\|^2_{H(\mathrm{div},\omega)} = \|.\|^2_{0,\omega} + \|\,\mathrm{div}\,.\|^2_{0,\omega}$, $H(\mathrm{div},\omega)$ is a Hilbert space. For any integer k, $P_k(\omega)$ is the set of polynomials of degree less than or equal to k.

2 Construction and analysis of the numerical scheme

Let Ω denote a bounded polygonal domain of $I\!R^2$. We consider the anisotropic diffusion problem :

$$\begin{cases} -\,\mathrm{div}(K\nabla u) = f & \text{over} \;\; \Omega, \\ u = 0 & \text{on} \;\; \Gamma, \end{cases} \tag{2.1}$$

with symmetric definite positive tensor K, assumed piecewise constant for simplicity, and $f \in L^2(\Omega)$. Let $(\mathcal{T}_h)$ be a family of triangulations of Ω, by triangles, regular in the usual finite element sense [5]. For all T in $\mathcal{T}_h$, there exist reals $d_{T,e}$ such that the bilinear form $a_T : (H^1(T))^2 \longrightarrow I\!R$ defined by

$$\forall p, q \in \left(H^1(T)\right)^2, \qquad a_T(p,q) = \sum_{e\in\partial T} d_{T,e} \left(\int_e p.n_T d\gamma\right)\left(\int_e q.n_T d\gamma\right).$$

verifies

$$\forall p, q \in (P_0(T))^2, \quad a_T(p,q) = \int_T K^{-1} p.q \, dx,$$

where n_T is the unit normal outward to T.

Remark 2.1. *We could give explicit expressions of the parameters $d_{T,e}$. Let T be a triangle with vertices a, b, c. The edge ab is denoted e and θ_e the opposite angle to e then,*

$$d_{T,e} = \frac{\sqrt{det(K)} < K^{-1}\overrightarrow{ac}, \overrightarrow{bc} >}{4meas(T)}$$

Clearly if $K = \alpha Id$ then $d_{T,e} = \dfrac{\alpha}{2} cotan(\theta_e)$.

In the sequel, we denote by $R_T x$ the unique element of $(P_0(T))^2$ checking

$$a_T(R_T x - x, q) = 0, \quad \forall q \in (P_0(T))^2 .$$

Let V_h be the nonconforming finite element space defined by

$$\begin{aligned} V_h = \{ & \lambda_h \in L^2(\Omega); \lambda_{h|_T} \in P_1(T), \\ & \forall T \in \mathcal{T}_h; \forall e \text{ interior edge}, \int_e [\lambda_h]_e \, d\gamma = 0 \text{ and } \forall e \subset \partial\Omega, \int_e \lambda_h d\gamma = 0 \} \end{aligned}$$

where $[\lambda_h]_e$ denotes the jump of the function λ_h across the edge e.

The non-standard finite element approximation we propose for the model problem is the following :

$$\begin{cases} \text{Find } \lambda_h \in V_h \text{ such that} \\ \forall \mu_h \in V_h, \quad \displaystyle\sum_{T \in \mathcal{T}_h} \int_T K \, \nabla\lambda_h . \nabla\mu_h = \sum_{T \in \mathcal{T}_h} \int_T f_T \left(\mu_h + \frac{1}{2}(x_G - R_T x).\nabla\mu_h \right) dx, \end{cases} \tag{2.2}$$

where $\forall T \in \mathcal{T}_h, \quad f_T = \dfrac{1}{\text{meas}(T)} \displaystyle\int_T f dx$ and x_G is the barycenter of T.

First of all, by adapting standard arguments used in the analysis of nonconforming finite element approximation of elliptic problems [5], we can easily prove that the discrete problem has a unique solution; moreover, if the weak solution u of the model problem belongs to the Sobolev space $H^{1+s}(\Omega)$ with $0 < s \leq 1$, then:

$$\left(\sum_{T \in \mathcal{T}_h} |u - \lambda_h|_{1,T}^2 \right)^{\frac{1}{2}} \leq C \left(h^s |u|_{1+s,\Omega} + \left(\sum_T h_T^2 \, \|f\|_{0,T}^2 + h_T^2 \, dist^2 \left(x_G, R_T x \right) \right)^{\frac{1}{2}} \right), \tag{2.3}$$

where h_T is the diameter of the triangle T.

Let us set

$$\forall T \in \mathcal{T}_h, \quad p_h = K.\nabla\lambda_h - f_T \frac{(x - R_T x)}{2} \quad \text{on } T.$$

The key point of the construction of the scheme is that p_h is an admissible field in the following sense :

Lemma 2.1 *The vector field p_h satisfies :*

$$p_h \in H(\mathrm{div}, \Omega) \quad \textit{and} \quad \forall T \in \mathcal{T}_h \quad -\mathrm{div}\, p_h = f_T \quad \textit{on } T.$$

Proof. It is obvious that $-\mathrm{div}\, p_h = f_T, \ \forall T \in \mathcal{T}_h$. Moreover if e is an interior edge of $\mathcal{T}_h$, $e = \partial T_1 \cap \partial T_2$, with $T_1, T_2 \in \mathcal{T}_h$ and $v_h^e \in V_h$ the associated basis function, i.e.,

$$\text{for any edge } \sigma \text{ of } \mathcal{T}_h, \quad \int_\sigma v_h^e d\gamma = \delta_\sigma^e, \quad \text{the Kronecker delta.}$$

Let us denote by $[p_h.n]_e$ the jump of the flux across the edge e. We have thus

$$\begin{aligned}
[p_h.n]_e = \int_e [p_h.n]_e v_h^e d\sigma &= \sum_{i=1}^{2} \int_{T_i} (p_h.\nabla v_h^e + v_h^e \,\mathrm{div}\, p_h)\, dx \\
&= \sum_{i=1}^{2} \int_{T_i} ((K.\nabla\lambda_h - f_T \frac{(x_G - R_T x)}{2}).\nabla v_h^e - f_T v_h^e) dx = 0,
\end{aligned}$$

which yields $p_h \in H(\mathrm{div}, \Omega)$.

In order to define the numerical scheme, we need to introduce some notations. Let λ_h be the solution of the discrete problem (2.2); For any $T \in \mathcal{T}_h$ and σ edge of T, we set:

$$F_{\sigma,T} = \int_\sigma p_h.n_T \, d\gamma; \quad u_{\sigma,T} = \frac{1}{\mathrm{meas}(\sigma)} \int_\sigma \lambda_h d\gamma$$

and

$$u_T = \frac{1}{\mathrm{meas}(T)} \int_T \Big(\lambda_h + \frac{1}{2}\nabla\lambda_h.(x_G - R_T x)\Big)\, dx + \frac{\rho_{T,h}^2}{4} f_T,$$

where

$$\rho_{T,h}^2 = \frac{1}{\mathrm{meas}(T)} a_T(x - R_T x, x - R_T x).$$

Lemma 2.2 *With the notations given above, one has the following scheme $\forall T, T_1, T_2 \in \mathcal{T}_h$,*

$$\left\{
\begin{aligned}
-\sum_{\sigma \in \partial T} F_{\sigma,T} &= \mathit{meas}(T)\, f_T \\
F_{\sigma,T_1} + F_{\sigma,T_2} &= 0, \quad \forall \sigma \in \partial T_1 \cap \partial T_2, \\
d_{T,\sigma} F_{\sigma,T} + u_T &= u_{\sigma,T}, \quad \forall \sigma \in \partial T, \\
u_{\sigma,T} &= 0, \qquad \mathit{if} \sigma \in \partial T \cap \partial\Omega.
\end{aligned}
\right. \tag{2.4}$$

Proof . First, we have

$$-\sum_{\sigma\in\partial T} F_{\sigma,T} = \text{meas}(T)\, f_T = \int_T f\,dx, \quad \forall T \in \mathcal{T}_h.$$

Indeed, we have :

$$-\int_T \operatorname{div} p_h dx = -\sum_{e\in\partial T}\int_\sigma p_h.n_T dx = \int_T f_T dx = \text{meas}(T)\, f_T.$$

And for any interior edge $\sigma \in \partial T_1 \cap \partial T_2$, $\quad F_{\sigma,T_1} + F_{\sigma,T_2} = 0$ is obvious since $p_h \in H(\text{div},\Omega))$.

Let $q_h \in RT_0(T) = (P_0(T))^2 + xP_0(T)$ such that $q_h.n_T|_\sigma = 1$ and $q_h.n_T|_e = 0, \forall \text{edge } e \neq \sigma$.

By one hand, we have,

$$B := a_T(p_h, q_h) - \int_{\partial T} \lambda_h q_h.n_T = d_{T,\sigma}\text{meas}(\sigma)\, F_{\sigma,T} - u_{\sigma,T}.\text{meas}(\sigma)$$

and on the other hand, if we set

$$A = a_T(\frac{\operatorname{div} p_h}{2}(x - R_T x), q_h),$$

and

$$\overline{\lambda_h} = \frac{1}{\text{meas}(T)}\int_T \lambda_h dx,$$

we get

$$a_T(p_h, q_h) - \int_{\partial T} \lambda_h q_h.n_T = a_T(p_h, q_h) - \int_T \nabla\lambda_h.q_h - \int_T \overline{\lambda_h} \operatorname{div} q_h dx$$

$$= a_T(p_h - \frac{\operatorname{div} p_h}{2}(x - R_T x), q_h) - \int_T \nabla\lambda_h.q_h + A - \int_T \overline{\lambda_h} \operatorname{div} q_h dx$$

$$= a_T(\nabla\lambda_h, q_h) - \int_T \nabla\lambda_h.q_h + A - \int_T \overline{\lambda_h} \operatorname{div} q_h dx$$

$$= a_T(\nabla\lambda_h, q_h - \frac{\operatorname{div} q_h}{2}(x - R_T x)) - \int_T \nabla\lambda_h.q_h + A - \int_T \overline{\lambda_h} \operatorname{div} q_h dx$$

$$= \int_T \left(q_h - \frac{\operatorname{div} q_h}{2} x - R_T x\right).\nabla\lambda_h - \int_T \nabla\lambda_h.q_h + A - \int_T \overline{\lambda_h} \operatorname{div} q_h dx$$

$$= -\int_T \left(\overline{\lambda_h} + \frac{1}{2}\nabla\lambda_h(x_G - R_T x)\right) \operatorname{div} q_h + A.$$

However, since $\int_T \operatorname{div} q_h = \int_{\partial T} q_h.n_T d\gamma = \text{meas}(\sigma)$, we obtain

$$B = -\text{meas}(\sigma)\left(\overline{\lambda_h} + \frac{1}{2}\nabla\lambda_h(x_G - R_T x)\right) + A$$

But we have also,

$$
\begin{aligned}
A &= a_T(\frac{\operatorname{div} p_h}{2}(x - R_T x), q_h) \\
&= a_T(\frac{\operatorname{div} p_h}{2}(x - R_T x), q_h - \frac{\operatorname{div} q_h}{2}(x - R_T x)) \\
&- \frac{\operatorname{div} q_h}{4} a_T(x - R_T x, x - R_T x).f_T \\
&= - \left(\int_T \operatorname{div} q_h dx \right) . \frac{\rho_{T,h}}{4}.f_T = -\text{meas}(\sigma)\, \frac{\rho_{T,h}}{4}.f_T.
\end{aligned}
$$

where

$$
\rho_{T,h} = \frac{1}{\text{meas}(T)} a_T(x - R_T x, x - R_T x)
$$

which implies

$$
B = \left(\overline{\lambda_h} + \frac{1}{2} \nabla \lambda_h (x_G - R_T x) + \frac{\rho_{T,h}}{4}.f_T \right)
$$

and thus scheme (2.4).

Using once more Lemma 2.1, we can derive the following a priori error estimate,

Lemma 2.3 *If the weak solution u of model problem* (2.1) *belongs to* $H^{1+s}(\Omega)$, $0 < s \leq 1$, *one has:*

$$
\left(\sum_{T \in \mathcal{T}_h} \|u - u_T\|_{0,T}^2 \right)^{\frac{1}{2}} \leq C \left(h^s |u|_{1+s,\Omega} + \left(\sum_T h_T^2 \|f\|_{0,T}^2 + h_T^2\, dist^2(x_G, R_T x) \right)^{\frac{1}{2}} \right). \tag{2.5}
$$

2.1 A focus on the treatment of an additional reaction term

Let us consider the problem of diffusion-reaction equations:

$$
\begin{cases}
-\operatorname{div}(\nabla u) + cu = f & \text{over } \Omega, \\
u = 0 & \text{on } \Gamma.
\end{cases} \tag{2.6}
$$

where $f \in L^2(\Omega)$, $c \in L^\infty(\Omega)$ with $c \geq 0$, and the following associated discrete problem :

$$
\begin{cases}
\text{Find } \lambda_h \in V_h \text{ such that} \\
\forall \mu_h \in V_h, \displaystyle\sum_{T \in \mathcal{T}_h} \int_T \nabla \lambda_h . \nabla \mu_h dx \\
\quad + \displaystyle\sum_{T \in \mathcal{T}_h} \alpha_T c_T \int_T \left(\overline{\lambda}_h + \frac{1}{2}(x_G - R_T x).\nabla \lambda_h \right) \left(\overline{\mu}_h + \frac{1}{2}(x_G - R_T x).\nabla \mu_h \right) dx \\
\quad = \displaystyle\sum_{T \in \mathcal{T}_h} \alpha_T \int_T f_T \left(\mu_h + \frac{1}{2}(x_G - R_T x).\nabla \mu_h \right) dx
\end{cases} \tag{2.7}
$$

where for all $T \in \mathcal{T}_h$,

$$c_T = \frac{1}{\text{meas}(T)} \int_T c dx \ , f_T = \frac{1}{\text{meas}(T)} \int_T f dx \ , \ \alpha_T = \frac{4}{4 + c_T \rho_T^2},$$

$\overline{\lambda}_h = \frac{1}{\text{meas}(T)} \int_T \lambda_h dx$ (and analogously for $\overline{\mu}_h$), and x_G is the barycenter of T.

Using the same arguments as before, we have in this case :

Lemma 2.4 *Let λ_h be the solution of the discrete problem. We introduce*

$$p_h = \nabla\lambda_h - \frac{2}{4 + c_T\rho_T^2}(f_T + c_T(\overline{\lambda}_h + \nabla\lambda_h(x_G - R_T x)))(x - R_T x).$$

$$F_{\sigma,T} = \int_\sigma p_h.n_T d\gamma, \qquad u_\sigma = \frac{1}{meas(\sigma)} \int_\sigma \lambda_h d\gamma$$

and

$$u_T = \frac{4}{4 + c_T\rho_T^2} \left(\frac{1}{meas(T)} \int_T \left(\lambda_h + \frac{1}{2}\nabla\lambda_h.(x_G - R_T x) \right) + \frac{\rho_{T,h}^2}{4} f_T \right),$$

where

$$\rho_{T,h}^2 = \frac{1}{meas(T)} a_T(x - R_T x, x - R_T x).$$

Then we have

$$\begin{aligned} -\sum_{\sigma\in\partial T} F_{\sigma,T} + c_T u_T &= meas(T) f_T \\ F_{\sigma,T_1} + F_{\sigma,T_2} &= 0, \quad \forall\sigma \in \partial T_1 \cap \partial T_2, \\ d_{T,\sigma} F_{\sigma,T} + u_T &= u_{\sigma,T}, \quad \forall\sigma \in \partial T, \\ u_{\sigma,T} &= 0, \qquad if \sigma \in \partial T \cap \partial\Omega. \end{aligned}$$

3 A posteriori error estimator for the diffusion model problem

Usually, error estimators for adaptive refinement require exact discrete solutions (see [10] and references therein), but in practical cases the exact solution is not available and so we are in the presence of solvers error. In this subsection, we introduce a posteriori error estimator for solutions obtained by black-box solver, in this case we are in the presence of many source of errors : *approximation, error solvers, post processing error …etc.* We indicate the a posteriori error estimator for the diffusion model equation:

$$\begin{cases} -\Delta u = f & \text{over } \ \Omega, \\ u = 0 & \text{on } \ \Gamma. \end{cases} \tag{3.1}$$

The solution is assumed to be obtained by any existing solver. The given estimator is valid also for equilibrium and mixed finite element approximations with or without numerical

integration.
Let $\mathcal{T}_h$ be a regular triangulation of Ω by triangles, E is the set of all edges and E_I the set of all interior edges. Given $T \in \mathcal{T}_h$, $\Delta(T)$ is the union of all elements of $\mathcal{T}_h$ sharing a vertex with T, ω_T is the union of all elements of $\mathcal{T}_h$ sharing an edge with T and E_T the set of all edges of T. We consider the finite dimensional space

$$V_h = \{v_h \in H^1(\Omega),\ \forall T \in \mathcal{T}_h \quad v_{h|T} \in \mathrm{P}_1(T)\} \tag{3.2}$$

$$E_h = \{p_h \in (L^2(\Omega))^2, \forall T \in \mathcal{T}_h \quad p_{h|T} \in \mathrm{RT}_0(T) = (P_0(T))^2 + xP_0(T)\} \tag{3.3}$$

and

$$M_h = \{v_h \in L^2(\Omega),\ \ \forall T \in \mathcal{T}_h \quad v_{h|T} \in \mathrm{P}_0(T)\} \tag{3.4}$$

Let $p_h \in E_h$ and $u_h \in M_h$, for all $T \in \mathcal{T}_h$ we set

$$\varepsilon_{1,T}(p_h) = \sup_{v_h \in V_h(T)} \frac{\int_{\Delta(T)} p_h \nabla v_h dx - \int_{\Delta(T)} v_h f dx}{|v_h|_{1,\Delta(T)}}, \tag{3.5a}$$

$$\varepsilon_{2,T}(p_h) = \sup_{\phi_h \in V_h(T)} \frac{\int_{\Delta(T)} p_h \operatorname{curl} \phi_h dx}{|\phi_h|_{1,\Delta(T)}}, \tag{3.5b}$$

$$\varepsilon_{3,T}(p_h, u_h) = \sup_{q_h \in E_h(T)} \frac{\int_{\omega_T} (p_h q_h + u_h \operatorname{div} q_h) dx}{\|q_h\|_{H(\mathrm{div},\omega_T)}}, \tag{3.5c}$$

$$\eta^2_{1,T}(p_h) = h_T^2 \|f - f_T\|^2_{0,T} + \sum_{l \in E_T} (h_l \|[p_h.t_l]_l\|^2_{0,l}), \tag{3.5d}$$

and

$$\eta_{2,T}(p_h) = h_T \|p_h\|_{0,T}, \tag{3.5e}$$

where

$$V_h(T) = \{v_h \in H^1_0(\Delta(T)), \qquad \forall T \in \Delta(T) \ \ v_{h|T} \in P_1(T)\}, \tag{3.6}$$

and

$$E_h(T) = \{q_h \in E_h \cap H(\mathrm{div}, \Omega), \qquad \forall T \notin \omega_T \ \ q_{h|T} = 0\}. \tag{3.7}$$

h_T and h_l are the diameters of T and l respectively. The outward normal to an edge l of some $T \in \mathcal{T}_h$ is written as $n_l = (n_{1,l}, n_{2,l})$ and we set $t_l = (n_{2,l}, -n_{1,l})$ for associated tangential direction. we denote by $[p_h.t_l]_l$ the jump of $p_h.t_l$ across the edge l.
In the sequel, C, C_1, C_2 are positive generic constants independent of h (which may change from one line to other).

Remark 3.1. *Let us notice that*

(1) Since

$$-\operatorname{div} p_h = \frac{1}{meas(T)} \int_T f dx = f_T \ \textit{on } T,$$

we have

$$\varepsilon_{1,T}^2(p_h) \le C \sum_{T\in\Delta(T)} h_T^2 \|f - f_T\|_{0,T}^2,$$

which is higher order perturbation of the error.

(2) *Let u_h^T, $\psi_h^T \in V_h(T)$ and $q_h^T \in E_h(T)$ be the unique solutions of the following respective problems*

$$(P1) \qquad \begin{cases} \text{Find } u_h^T \in V_h(T) \text{ such that} \\ \forall v_h \in V_h(T), \quad \int_{\Delta(T)} \nabla u_h^T \nabla v_h dx = \int_{\Delta(T)} p_h \nabla v_h dx - \int_{\Delta(T)} f v_h dx, \end{cases}$$

$$(P2) \qquad \begin{cases} \text{Find } \psi_h^T \in V_h(T) \text{ such that} \\ \forall \phi_h \in V_h(T), \quad \int_{\Delta(T)} \operatorname{curl} \psi_h^T \operatorname{curl} \phi_h dx = \int_{\Delta(T)} p_h \operatorname{curl} \phi_h dx \end{cases}$$

$$(P3) \qquad \begin{cases} \text{Find } q_h^T \in E_h(T) \text{ such that} \\ \forall s_h \in E_h(T), \quad \int_{\omega_T} (q_h^T s_h + \operatorname{div} q_h^T \operatorname{div} s_h) dx = \int_{\omega_T} (p_h s_h + u_h \operatorname{div} s_h) dx. \end{cases}$$

It is easy to see that

$$|u_h^T|_{1,T} = \varepsilon_{1,T}(p_h) \qquad |\psi_h^T|_{1,T} = \varepsilon_{2,T}(p_h) \quad \text{and} \quad \|q_h^T\|_{H(\operatorname{div},\omega_T)} = \varepsilon_{3,T}(p_h, u_h).$$

We have the following error estimates,

Theorem 3.1 *Let $u \in H_0^1(\Omega)$ be the weak solution of the model problem (3.1), $p = \nabla u$, $p_h \in E_h$ and $u_h \in M_h$ the solution of the given numerical scheme. Then there exists a positive constant C only depending on the minimum angle of $\mathcal{T}_h$ such that*

$$\|u - u_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \le C \Big\{ \Big(\sum_{T\in\mathcal{T}_h} (\eta_{1,T}^2(p_h) + \varepsilon_{1,T}^2(p_h) + \varepsilon_{2,T}^2(p_h)) \Big)^{\frac{1}{2}}$$

$$+ \Big(\sum_{T\in\mathcal{T}_h} (\eta_{2,T}^2(p_h) + \varepsilon_{3,T}^2(p_h, u_h)) \Big)^{\frac{1}{2}} \Big\}.$$

Moreover, we have

$$\sum_{i=1}^2 \eta_{i,T}(p_h) + \sum_{i=1}^2 \varepsilon_{i,T}(p_h) + \varepsilon_{3,T}(p_h, u_h) \le C_1(\|p - p_h\|_{0,\Delta(T)} + \|u - u_h\|_{0,\Delta(T)})$$

$$+ C_2 \Big(\sum_{T'\in\omega_T} h_{T'}^2 \|f - f_{T'}\|_{0,T'}^2 \Big)^{\frac{1}{2}}$$

Proof : First, Using Helmholtz-decomposition, we have $e_h = p - p_h = \nabla w + \operatorname{curl}\zeta$, with $w \in H_0^1(\Omega)$, $\zeta \in H^1(\Omega)$ and $\int_\Omega \nabla w \operatorname{curl}\zeta dx = 0$.
Let us remark that the orthogonality implies the following error decomposition :

$$\|e_h\|_{0,\Omega}^2 = |w|_{1,\Omega}^2 + \|\operatorname{curl}\zeta\|_{0,\Omega}^2, \tag{3.8}$$

$$|w|_{1,\Omega}^2 = \int_\Omega e_h \nabla w dx \quad \text{and} \quad \|\operatorname{curl}\zeta\|_{0,\Omega}^2 = \int_\Omega e_h \operatorname{curl}\zeta dx. \tag{3.9}$$

Now, let $w^I \in V_h$ and $\zeta^I \in V_h$ be continuous approximations of w and ζ respectively such that :

$$\forall T \in \mathcal{T}_h, \quad \|w - w^I\|_{0,T} \leq C\text{meas}(T)^{\frac{1}{2}}|w|_{1,\Delta(T)}, \tag{3.10a}$$

$$|w^I|_{1,\Omega} \leq C|w|_{1,\Omega}, \tag{3.10b}$$

and

$$\forall l \in E_T, \quad \|w - w^I\|_{0,l} \leq C\text{meas}(l)^{\frac{1}{2}}|w|_{1,\Delta(l)}, \tag{3.10c}$$

(and analogously for ζ) where $\Delta(l)$ is the union of the elements T sharing l. Moreover we assume that the interpolation preserves boundary conditions, that is, $w^I \in V_h \cap H_0^1(\Omega)$. It is well known that such approximations exist (see [5], [8]).
First, according to (3.9) we have by element-wise integration by parts, noting that $w - w^I \in H_0^1(\Omega)$,

$$\|\nabla w\|_{0,\Omega}^2 = \int_\Omega e_h \nabla(w - w^I) dx + \int_\Omega f w^I dx - \int_\Omega p_h \nabla w^I dx$$

$$= \sum_{T\in\mathcal{T}} \int_T (f + \operatorname{div} p_h)(w^I - w) dx + \int_\Omega f w^I dx - \int_\Omega p_h \nabla w^I dx.$$

From Cauchy's inequality and from (3.10a) and (3.10c),

$$\sum_{T\in\mathcal{T}_h} \int_T (f + \operatorname{div} p_h)(w^I - w) dx \leq \Big(\sum_{T\in\mathcal{T}_h} h_T^2 \|f + \operatorname{div} p_h\|_{0,T}^2\Big)^{\frac{1}{2}} |w|_{1,\Omega}.$$

Using (3.10b), we obtain

$$\int_\Omega f w^I dx - \int_\Omega p_h \nabla w^I dx \leq C\Big(\sum_{T\in\mathcal{T}_h} \varepsilon_{1,T}^2(p_h)\Big)^{\frac{1}{2}} |w^I|_{1,\Omega} \leq C\Big(\sum_{T\in\mathcal{T}_h} \varepsilon_{1,T}^2(p_h)\Big)^{\frac{1}{2}} |w|_{1,\Omega}.$$

Then we have

$$\|\nabla w\|_{0,\Omega} \leq C\Big(\sum_{T\in\mathcal{T}_h} h_T^2 \|f + \operatorname{div} p_h\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \varepsilon_{1,T}^2(p_h)\Big)^{\frac{1}{2}}. \tag{3.11}$$

Arguing as above, since $p = \nabla u$ and $p_h \in E_h$, by element-wise integration by parts we have

$$\int_\Omega e_h \operatorname{curl}(\zeta - \zeta^I) dx = \sum_{l\in E} \int_l (\zeta - \zeta^I)[p_h.t_l]_l d\sigma.$$

By Cauchy's inequality (3.10c) we obtain

$$\int_\Omega e_h \operatorname{curl}(\zeta - \zeta^I)dx \le C\Big(\sum_{l\in E} h_l \|[p_h.t_l]_l\|^2_{0,l}\Big)^{\frac{1}{2}} \|\nabla\zeta\|_{0,\Omega}. \tag{3.12}$$

Finally, since $\|\operatorname{curl}\zeta\|_{0,\Omega} = \|\nabla\zeta\|_{0,\Omega}$,

$$\int_\Omega e_h \operatorname{curl}\zeta^I dx = -\int_\Omega p_h \operatorname{curl}\zeta^I dx \tag{3.13}$$

and

$$\|\operatorname{curl}\zeta\|^2_{0,\Omega} = \int_\Omega e_h \operatorname{curl}\zeta dx = \int_\Omega e_h \operatorname{curl}(\zeta - \zeta^I)dx + \int_\Omega e_h \operatorname{curl}\zeta^I dx,$$

and using (3.12) and (3.10b), we obtain

$$\|\operatorname{curl}\zeta\|_{0,\Omega} \le C\Big(\sum_{l\in E} h_l\|[p_h.t_l]_l\|^2_{0,l} + \sum_{T\in\mathcal{T}_h} \varepsilon^2_{2,T}(p_h)\Big)^{\frac{1}{2}}. \tag{3.14}$$

Using the Helmholtz decomposition (3.8) together with the estimates (3.11) and (3.14), we get

$$\|p - p_h\|^2_{0,\Omega} \le C \sum_{T\in\mathcal{T}_h} (\eta^2_{1,T}(p_h) + \varepsilon^2_{1,T}(p_h) + \varepsilon^2_{2,T}(p_h)) \qquad (3.15).$$

Now, let $P_h u \in M_h$ defined by

$$\forall T \in \mathcal{T}_h,\ P_h u = \frac{1}{\operatorname{meas}(T)} \int_T u dx \quad \text{on } T.$$

We have

$$\|u - P_h u\|_{0,T} \le C h_T \|p\|_{0,T} \le C(h_T\|p - p_h\|_{0,T} + h_T\|p_h\|_{0,T}). \tag{3.16}$$

On the other hand, using the inf-sup condition we have

$$\|u_h - P_h u\|_{0,\Omega} \le C \sup_{q_h\in E_h} \frac{\int_\Omega (u_h - P_h u)\operatorname{div} q_h dx}{\|q_h\|_{H(\operatorname{div},\Omega)}},$$

and since

$$\int_\Omega (u_h - P_h u)\operatorname{div} q_h dx = \int_\Omega (u_h \operatorname{div} q_h dx + p_h q_h)dx + \int_\Omega (p - p_h) q_h dx$$

we obtain easily that

$$\|u_h - P_h u\|_{0,\Omega} \le C\Big(\sum_{T\in\mathcal{T}_h} \varepsilon^2_{3,T}(p_h, u_h)\Big)^{\frac{1}{2}} + \|p - p_h\|_{0,\Omega}. \qquad (3.17).$$

By triangular inequality, and using (3.15) and (3.17), we have :

$$\|u-u_h\|_{0,\Omega}+\|p-p_h\|_{0,\Omega}\leq C\Big\{\Big(\sum_{T\in\mathcal{T}_h}(\eta_{1,T}^2(p_h)+\varepsilon_{1,T}^2(p_h)+\varepsilon_{2,T}^2(p_h))\Big)^{\frac{1}{2}}$$

$$+\Big(\sum_{T\in\mathcal{T}_h}(\eta_{2,T}^2(p_h)+\varepsilon_{3,T}^2(p_h,u_h))\Big)^{\frac{1}{2}}\Big\}.$$

To indicate the efficiency of the a posteriori error estimator we follow Verfürth [10] and show a local reverse up to higher order perturbations.
For each $T\in\mathcal{T}_h$, we reset :

$$\omega_T=\{T'\in\mathcal{T}_h \text{ such that } T \text{ and } T' \text{ have a common edge}\},$$

$$\forall T\in\mathcal{T}_h,\quad f_T=\frac{1}{\mathrm{meas}(T)}\int_T f\,dx,$$

and for all $l\in E_I$, we denote by T_+ and T_- the two elements of $\mathcal{T}_h$ sharing this edge. Let b_T be the standard bubble function on T with $\max_T b_T=1$, as defined in [10]. Then norms $\|.\|_{0,T}$ and $\|b_T.\|_{0,T}$ are equivalent on $P_0(T)$, and

$$\|r_T\|_{0,T}^2\leq C\int_T b_Tr_T(f_T-f+\mathrm{div}(p_h-p))dx\leq C\int_T\nabla(b_Tr_T).(p_h-p)dx+C\|r_T\|_{0,T}\|f-f_T\|_{0,T}$$

where $r_T:=f_T+\mathrm{div}\,p_h$ on T. Then we have

$$\|r_T\|_{0,T}^2\leq C|b_Tr_T|_{1,T}\|p-p_h\|_{0,T}+C\|r_T\|_{0,T}\|f-f_T\|_{0,T},$$

Using the inverse estimate $|r_Tb_T|_{1,T}\leq Ch_T^{-1}\|r_T\|_{0,T}$, we obtain

$$h_T\|f_T+\mathrm{div}\,p_h\|_{0,T}\leq C_1\|p-p_h\|_{0,T}+C_2h_T\|f-f_T\|_{0,T}. \tag{3.18}$$

Concerning the jump terms for $l\in E_T\cap E_I$, let b_l be the standard bubble function on T vanishing on $\partial T\backslash l$ such that $\max_T b_l=1$ (see [10]). Then again the norms $\|.\|_{0,l}$ and $\|b_l.\|_{0,l}$ are equivalent on $P_1(l)$. Let $l\in E_I$, then using the extension operator $P:\mathcal{C}^0(l)\longrightarrow\mathcal{C}^0(T_+\cup T_-)$ of [10], it follows that

$$\|[p_h.t_l]_l\|_{0,l}^2\leq C\int_l b_lP([p_h.t_l]_l)[p_h.t_l]_ld\sigma=C\int_{T_+\cup T_-}p_h\,\mathrm{curl}(b_lP([p_h.t_l]_l))dx.$$

Because of the inverse inequality

$$\|b_lP([p_h.t_l]_l)\|_{1,T_+\cup T_-}\leq Ch_l^{-1}\|b_lP([p_h.t_l]_l)\|_{0,T_+\cup T_-},$$

the equality

$$\int_{T_+\cup T_-}p\,\mathrm{curl}(b_lP([p_h.t_l]_l))dx=0,$$

using Cauchy's inequality and $\|b_lP([p_h.t_l]_l)\|_{0,T_+\cup T_-}\leq Ch_l^{\frac{1}{2}}\|[p_h.t_l]_l\|_{0,l}$, we obtain

$$h_l^{\frac{1}{2}}\|[p_h.t_l]_l\|_{0,l}\leq C\|p-p_h\|_{0,T_+\cup T_-}. \tag{3.19}$$

Finally, it is easy to see that

$$\forall i = 1, 2 \qquad \varepsilon_{i,T}(p_h) \leq \|p - p_h\|_{0,\Delta(T)}. \tag{3.20}$$

Combining (3.18)-(3.20) we obtain

$$\eta_{1,T}^2(p_h) + \varepsilon_{1,T}^2(p_h) + \varepsilon_{2,T}^2(p_h) \leq C_1 \sum_{T' \in \Delta(T)} \|p - p_h\|_{0,T'}^2 + C_2 \sum_{T' \in \omega_T} h_{T'}^2 \|f - f_{T'}\|_{0,T'}^2. \tag{3.21}$$

Introducing as above the bubble function b_T, we get

$$\|p_h\|_{0,T}^2 \leq C \int p_h b_T p_h dx = C \int_T b_T p_h (p_h - p) dx + C \int_T \nabla u.(b_T p_h). \tag{3.22}$$

Since $\int_T \operatorname{div}(b_T p_h) dx = 0$ and $u_h \in M_h$, by element-wise integration by parts we have

$$\int_T \nabla u.(b_T p_h) dx = \int_T (u_h - u) \operatorname{div}(b_T p_h) dx. \tag{3.23}$$

Using the inverse inequality $\|\operatorname{div}(b_T p_h)\|_{0,T} \leq C h_T \|p_h\|_{0,T}$, and (3.22)-(3.23), we obtain

$$\eta_{2,T}(p_h) \leq C(\|p - p_h\|_{0,T} + \|u - u_h\|_{0,T}). \tag{3.24}$$

Finally, it is easy to check that

$$\varepsilon_{3,T}(p_h, u_h) \leq C(\|p - p_h\|_{0,\Delta(T)} + \|u - u_h\|_{0,\Delta(T)}). \tag{3.25}$$

Using the estimates (3.24)-(3.25), we get

$$\forall T \in \mathcal{T}_h, \quad \eta_{2,T}(p_h) + \varepsilon_{3,T}(p_h, u_h) \leq C(\|p - p_h\|_{0,\Delta(T)} + \|u - u_h\|_{0,\Delta(T)}). \tag{3.26}$$

Finally, using (3.21) and (3.26) concludes the proof.

References

[1] R.A. Adams. Sobolev Spaces. Academic Press, New York (1975).

[2] F. Brezzi, M. Fortin and L.D. Marini. Error analysis of piecewise constant pressure approximations of Darcy's law. Comput. Methods Appl. Mech. Engrg. 195 , no. 13-16, pp. 1547–1559 (2006).

[3] C. Carstensen and R.H.W. Hoppe. Error reduction and convergence for an adaptive mixed finite element method. Math. Comp. Vol 75, No 255, pp. 1033-1042 (2006)

[4] C. Carstensen and R.H.W. Hoppe. Convergence analysis of an adaptive nonconforming finite element method. Numer. Math. Vol 103, No 2, pp. 251–266 (2006)

[5] P. G. Ciarlet. Handbook of numerical analysis. Vol. II. Eds. Ciarlet, P. G. and Lions, J.-L.. Finite element methods. Part 1. North-Holland, Amsterdam (1991).

[6] R. Eymard, T. Gallouët and R. Herbin. Finite volume methods. Handbook of numerical analysis, Vol. VII, pp 713–1020, North-Holland,Amsterdam 2000.

[7] R. Eymard, T. Gallouët and R. Herbin. A finite volume scheme for anisotropic diffusion problems. C.R. Acad. Sci. Paris, Ser I 339 (2004).

[8] Scott L. Ridgway and Zhang Shangyou . Finite element interpolation of non smooth functions satisfying boundary conditions . Math. Comp., Vol 54, No 190, pp 483-493(1990)

[9] William J. Layton, Friedhelm Schieweck and Ivan Yotov. Coupling fluid flow with porous media flow. SIAM J. Numer. Anal. Vol 40, No 6, pp. 2195-2218 (2003).

[10] R. Verfürth. A review of a posteriori error estimation and adaptive mesh-refinement techniques.Wiley Teubner, 3 (1996)